Effective action in DSR1 quantum field theory

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Abstract

We present the one-loop effective action of a quantum scalar field with DSR1 space-time symmetry as a sum over field modes. The effective action has real and imaginary parts and manifest charge conjugation asymmetry, which provides an alternative theoretical setting to the study of the particle-antiparticle asymmetry in nature.

We report here a recent result concerning the one-loop effective action of doubly-relativistic theory of type 1 (DSR1) [1, 2]. To set the stage for this result we start by discussing analogous results in usual relativistic theory described by the Poincaré algebra. Then we consider the κ-deformed Poincaré algebra in standard basis and in bicrossproduct basis [3, 4, 5, 6]. The latter is the mathematical setting of DSR1 theory [2].

It is usually assumed that local space-time symmetries are completely described by the Poincaré algebra, whose defining commutation relations between the basis elements we exhibit here with the purpose of comparison with the deformed algebras that we consider in the present work. In usual notation we have

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \ , \\ i[P^\mu, J^{\rho\sigma}] &= g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho \ , \\ i[J^{\mu\nu}, J^{\rho\sigma}] &= g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} \ , \end{aligned} \tag{1}$$

where we identify P^0 as the energy, P^i ($i = 1, 2, 3$) as the components of momentum three-vector, $J^i = \varepsilon^{ijk} J^{jk}/2$ ($i = 1, 2, 3$) as the components of angular-momentum three-vector and $K^i = J^{i0}$ ($i = 1, 2, 3$) as the components of the boost three-vector. The first Casimir invariant of this algebra is the relation between energy and momentum given by

$$P_\mu P^\mu = \mathbf{P}^2 - P_0^2 = -m^2 \ , \tag{2}$$

where m^2 is a positive real scalar which labels the irreducible representation of the algebra under consideration. For this case m is the mass of the field excitation, *i.e.*, its

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rest frame energy. It leads to the usual relativistic dispersion relations $p^0 = \omega^{(\pm)}(\mathbf{p})$, where

$$\omega^{(\pm)}(\mathbf{p}) = \pm\sqrt{\mathbf{p}^2 + m^2}\ . \tag{3}$$

The equality $\omega^{(+)}(\mathbf{p}) = -\omega^{(-)}(\mathbf{p})$ describes charge conjugation symmetry, *i.e*, particles and antiparticles have the same dispersion relation (except for the signs).

From the Casimir invariant (2) we also identify the Schwinger proper-time Hamiltonian for a relativistic scalar field [7],

$$H = \mathbf{P}^2 - P_0^2 + m^2\ . \tag{4}$$

For simplicity we will be considering here only the case of a scalar field, which we represent by ϕ.

The proper-time Hamiltonian is all we need in order to calculate the one-loop effective action given the in Schwinger representation, which has the form [7]

$$\mathcal{W} = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\ Tr\ e^{-isH} \tag{5}$$

where the real variable s is the so-called proper time, H is the proper-time Hamiltonian of the theory and Tr represents the total trace on the exponential operator. The one-loop effective action $\mathcal{W}$ satisfies

$$\langle 0\, t_2 | 0\, t_1 \rangle = e^{i\mathcal{W}} \tag{6}$$

where $\langle 0\, t_2 | 0\, t_1 \rangle$ is the vacuum persistence amplitude from an instant t_1 in the remote past to an instant t_2 in the distant future. In (6) we identify the vacuum energy E_0 as given by the real part of the effective action,

$$E_0 = -\frac{\Re\mathcal{W}}{T}\ , \tag{7}$$

where $T = t_2 - t_1$. The shift of this energy caused by external conditions is the Casimir energy [8, 9], an important observable in any relativistic field theory. The properly renormalized imaginary part of the effective action gives the probability of vacuum decay,

$$1 - |\langle 0_{t_2} | 0_{t_1} \rangle|^2 = 1 - e^{-2\Im\mathcal{W}}\ . \tag{8}$$

In this way, a positive imaginary contribution coming from the effective action may give rise to the phenomenon of creation of field excitations. Obviously enough, a negative imaginary part would lead to the inconsistent result of a negative probability. It is remarkable that two genuine and important quantum field effects, such as the Casimir energy and vacuum decay, can be obtained from the sole information of what is the Casimir invariant of the algebra of space-time symmetries of the field. From this algebraic invariant we also obtain the expression of the vacuum energy as the half sum of field frequencies. Indeed, by substituting into (5) the proper-time Hamiltonian H given in (4) we arrive at

$$\frac{\mathcal{W}}{T} = -\sum_{\mathbf{p}} \frac{1}{2}|\omega^{(\pm)}(\mathbf{p})|\ , \tag{9}$$

where $\omega^{(\pm)}(\mathbf{p})$ is given by the usual dispersion relations (3). The identification of the half sum of frequencies in (9) with the vacuum energy then follows from the expression (7) of the effective action in terms of the vacuum energy.

Results in quantum gravity and string theory, as well as some experimental astrophysical paradoxes, seem to indicate the existence of new dispersion relations which differ from the usual relativistic relations (3) [2, 10]. Those new relations depend on a dimensionful parameter that set the scale at which the usual relation fails and the new ones should be considered. In a certain limit of the parameter the new relations reduce to the usual one, due to the fact that at this limit we are back to the regime of validity of Poicaré invariance. In this way the new relations can be mathematically described as deformations of the usual one and the dimensionful parameter plays the role of a deformation parameter. The proper way of obtaining those deformed dispersion relations is from Casimir invariants of deformations of Poincaré algebras. We are going to deal with the κ-deformed Poincaré algebras [3, 4, 5, 6], as the most promising mathematical structure to describe possible new space-time symmetries. Those κ-deformed Poincaré algebras are examples of Hopf algebras also known in the literature as quantum algebras or quantum groups (cf., e.g, [11]). Since we are not concerned with the co-algebraic part of such quantum groups here we exhibit only its algebraic parts.

The original κ-deformed Poincaré algebra [3, 4] is defined in the so called standard basis [6] and is given by

$$
\begin{gathered}
[P^\mu, P^\nu] = 0 \ , \\
\left[J^i, J^j\right] = i\,\varepsilon^{ijk}\, J^k \ , \qquad \left[J^i, P^j\right] = i\,\varepsilon^{ijk}\, P^k \ , \qquad \left[J^i, P^0\right] = 0 \ , \\
\left[K^i, K^j\right] = -i\,\varepsilon^{ijk}\, \left(J^k\, \cosh(P^0/\kappa) - (P^k/4\kappa^2)\, \mathbf{P}\cdot\mathbf{J}\right) \ , \\
\left[K^i, P^j\right] = i\,\delta^{ij}\,\kappa\,\sinh(P^0/\kappa) \ , \\
\left[K^i, P^0\right] = i\,P^i \ , \qquad \left[J^i, K^j\right] = -i\,\varepsilon^{ijk}\, K^k \ ,
\end{gathered}
\tag{10}
$$

where κ is a real positive deformation parameter with dimension of mass. This parameter sets for energy, length and time the respective scales $E_\kappa = \kappa c^2$, $\ell_\kappa = \hbar/(\kappa c)$ and $t_\kappa = \hbar/(\kappa c^2)$. The Casimir invariant relating energy and momentum in this algebra is given by

$$
\mathbf{P}^2 - \left(2\kappa \sinh\frac{P^0}{2\kappa}\right)^2 = -m^2 \ , \tag{11}
$$

which leads immediately to the following dispersion relations

$$
p^0 = \omega^{(\pm)}(\mathbf{p}) = \pm 2\kappa \sinh^{-1}\left(\frac{1}{2\kappa}\sqrt{\mathbf{p}^2 + m^2}\right) \ . \tag{12}
$$

In this theory, that we call for short κ-deformed theory, the proper-time Hamiltonian is

$$
H_\kappa = \mathbf{P}^2 - (2\kappa)^2 \sinh^2\left(\frac{P_o}{2\kappa}\right) + m^2 \ . \tag{13}
$$

In the limit in which the deformation parameter κ goes to infinity the κ-deformed Poincaré algebra (10) reduces to the usual Poincaré algebra (1). In this limit the κ-deformed Casimir invariant (11), dispersion relations (12) and proper-time Hamiltonian

(13) also reduce to the corresponding non-deformed quantities given in (2), (3) and (4), respectively.

By inserting this κ-deformed proper time Hamiltonian (13) into (5), we obtain the following expression for the κ-deformed effective action

$$\frac{\mathcal{W}_\kappa}{T} = -\sum_{\mathbf{p}} \frac{1}{2}|\omega^{(\pm)}(\mathbf{p})| + i\frac{1}{2\pi\kappa}\sum_{\mathbf{p}} \frac{1}{2}|\omega^{(\pm)}(\mathbf{p})|^2 \ , \tag{14}$$

where now $\omega^{(\pm)}(\mathbf{p})$ is obtained from the κ-deformed dispersion relation (12). The real part has the usual form of half sum of field frequencies as in the non-deformed case, although the frequencies which are summed here are the κ-deformed (12). The imaginary part has the interesting property of being proportional to the sum of the squares of the κ-deformed field frequencies. If the scalar field is submitted to Dirichlet boundary conditions on parallel plates of side ℓ and separation a, we obtain from the real part of the κ-deformed effective action the Casimir energy [12]

$$\mathcal{E}(a) = -\frac{\ell^2}{16\pi^2 a^3}\sum_{n=1}\int_0^\infty d\sigma\sigma e^{-n^2\sigma-(2\kappa^2+m^2)a^2/\sigma}\sqrt{\frac{4a^2\kappa^2}{\pi\sigma}}\pi \mathrm{I}_0\left(\frac{2a^2\kappa^2}{\sigma}\right) \tag{15}$$

and from the imaginary part the following creation rate of field excitations [12]

$$\frac{\Im\mathcal{W}_\kappa}{T a\ell^2} = \frac{1}{16\pi^2 a^4}\sum_{n=1}\int_0^\infty d\sigma\sigma e^{-n^2\sigma-(2\kappa^2+m^2)a^2/\sigma}\sqrt{\frac{4a^2\kappa^2}{\pi\sigma}}\mathrm{K}_0\left(\frac{2a^2\kappa^2}{\sigma}\right) \ , \tag{16}$$

where I_0 and K_0 are the modified Bessel functions. It is important to note the manifest positiveness of the renormalized imaginary part (16) of the effective action. The κ-deformed Casimir energy (11) reduces to the Casimir energy of the usual scalar field in the limit $\kappa \to \infty$ in which the deformation disappears. The creation rate goes to zero in this limit or in the limit $a \to \infty$ of infinite separation of the plates. In this way the creation rate is a consequence of both deformation and boundary condition; if one of those factors is not present the creation rate vanishes.

Let us now turn to the κ-deformation of the Poincaré algebra in the so called bicrossproduct basis, which is given by [5, 6]

$$\begin{gathered}
[P_\mu, P_\nu] = 0 \\
[J_i, P_j] = i\epsilon_{ijk}P_k \qquad [J_i, P_0] = 0 \\
[K_i, P_j] = i\delta_{ij}\left((1/2\lambda)(1-e^{2P_0\lambda}) + (\lambda/2)\mathbf{P}^2\right) - i\lambda P_i P_j \\
[K_i, P_0] = iP_i \\
i[J_{\mu\nu}, J_{\rho\sigma}] = g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho} + g_{\mu\sigma}J_{\nu\rho}
\end{gathered} \tag{17}$$

where $\lambda = 1/\kappa$ is an alternative parameter with the dimension of length. The doubly special relativity of type 1 is a theory whose space-time symmetry is described by this algebra [1, 2]. This parameter provides the second invariant, besides the speed of light, which characterizes the theory as doubly special relativity. In DSR1 there is a natural

scale for length $\ell_\lambda = \lambda$ and also scales for energy and time, namely $E_\lambda = \hbar c/\lambda$ and $t_\lambda = \lambda/c$. In most applications ℓ_λ is taken as the Planck length. For brevity we refer to (17) as the λ-deformed Poincaré algebra. In the limit $\lambda \to 0$ this deformed algebra reduces to the Poincaré algebra.

From the λ-deformed Poincaré algebra (17) we have the following Casimir invariant involving energy and momentum

$$\mathbf{P}^2 e^{\lambda P_0} - \frac{e^{\lambda P_0} + e^{-\lambda P_0} - 2}{\lambda^2} = -m^2 \ , \tag{18}$$

Let us take from this Casimir invariant the following proper-time Hamiltonian for a scalar field

$$H_{\lambda *} = \mathbf{P}^2 e^{\lambda P^0} + m^2 - \frac{1}{\lambda^2}\left(e^{\lambda P^0} - 2 + e^{-\lambda P^0}\right) . \tag{19}$$

From the Casimir invariant (18) we also identify the λ-deformed dispersion relations $p^0 = \omega^{(\pm)}(\mathbf{p})$, where

$$\omega^{(+)}(\mathbf{p}) = -\frac{1}{\lambda}\log\left[1 + \frac{\lambda^2 m^2}{2} - \sqrt{\left(\frac{\lambda^2 m^2}{2}\right)^2 + \lambda^2\left(\mathbf{p}^2 + m^2\right)}\right] \tag{20}$$

are positive frequencies defined for $|\mathbf{p}| < 1/\lambda$ and

$$\omega^{(-)}(\mathbf{p}) = -\frac{1}{\lambda}\log\left[1 + \frac{\lambda^2 m^2}{2} + \sqrt{\left(\frac{\lambda^2 m^2}{2}\right)^2 + \lambda^2\left(\mathbf{p}^2 + m^2\right)}\right] \tag{21}$$

are negative frequencies defined for any $|\mathbf{p}|$. Notice that for $|\mathbf{p}| > 1/\lambda$ expression (20) for the positive frequency is not a real number and for this reason it should be discarded. However, we will see that its real part

$$\omega^{(>)}(\mathbf{p}) = -\frac{1}{\lambda}\log\left[-1 - \frac{\lambda^2 m^2}{2} + \sqrt{\left(\frac{\lambda^2 m^2}{2}\right)^2 + \lambda^2\left(\mathbf{p}^2 + m^2\right)}\right] \ , \tag{22}$$

for $|\mathbf{p}| > 1/\lambda$, appears naturally in the development of the formalism due to the fact that the Casimir invariant (18) was taken unrestrictedly to define the proper-time Hamiltonian (19) of the theory. The theory arising from those dispersion relations has no invariance under charge conjugation. In particular, particles have the saturation momentum $p_c = 1/\lambda$ while antiparticles have no constraint on its momentum values. Let us call the momentum or any quantity thereof depending unsaturated or transaturated whenever the magnitude of the momentum is lower or greater than the saturation momentum $1/\lambda$, respectively.

As an alternative way of defining a theory from the Casimir invariant (18) it is possible to use only the dispersion relation $p^0 = \pm\omega^{(+)}(\mathbf{p})$ with charge conjugation symmetry imposed on the theory. In this case both particles and antiparticles have saturation momentum $1/\lambda$. We will consider the theory with charge conjugation asymmetry derived from the proper-time Hamiltonian (19). To distinguish it from other

possibilities we refer to it as DSR1* theory. The main result that we present for the DSR1* theory is a calculation of its one-loop effective action leading to an expression in terms of sum of field modes.

The calculation of the effective action for DSR1* theory is a rather involved problem, due to its most essential features, namely: the lack of charge conjugation symmetry in the theory and the existence of a saturation momentum $1/\lambda$, which leads to a rather peculiar distribution of poles in the plane of complex frequencies. For simplicity we consider a two-dimensional space-time. Although not necessary for our purposes here, it is convenient for future applications to impose periodic boundary conditions on the scalar field ϕ, $\phi(x+a,t) = \phi(x,t)$, where a is the periodicity length. In this case momentum is discretized as $p^1 = 2\pi n/a$ ($n \in \mathbb{Z}$) and the saturation momentum becomes $[a/(2\pi\lambda)]$, the greatest integer which is smaller than $a/(2\pi\lambda)$. After a long calculation which starts by substituting the DSR1* proper-time Hamiltonian (19) in the Schwinger's formula (5) we obtain the following expression for the one-loop DSR1* effective action

$$\begin{aligned}\frac{\mathcal{W}_\lambda}{T} = &-\frac{1}{2}\sum_{|n|\le\left[\frac{a}{2\pi\lambda}\right]}|\omega_n^{(-)}| - \frac{1}{4}\sum_{|n|>\left[\frac{a}{2\pi\lambda}\right]}|\omega_n^{(-)}| \\ &+i\,\frac{\lambda}{2\pi}\left[\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\omega_n^{(-)}\right)^2 + \frac{1}{2}\sum_{|n|\le\left[\frac{a}{2\pi\lambda}\right]}\left(\omega_n^{(+)}\right)^2 + \frac{1}{2}\sum_{|n|>\left[\frac{a}{2\pi\lambda}\right]}\left(\omega_n^{(>)}\right)^2\right], \quad (23)\end{aligned}$$

where $\omega_n^{(+)}$, $\omega_n^{(-)}$ and $\omega_n^{(>)}$ are the discretized versions of the frequencies defined in (20), (21) and (22), respectively. By comparing this expression for the DSR1* action with the corresponding expression (14) for the κ-deformed action it is evident that the former has a more complicated structure that seems to be a result of the asymmetry between the particle and antiparticle sectors and of the existence of a saturation momentum. The expression (23) for the effective action as a sum of modes is well suited for physical interpretation, to which we proceed now.

First of all, the DSR1* effective action has real and imaginary parts, as in the κ-deformed case, and the whole expression has the correct limit when the deformation disappears, *i.e.*, when $\lambda \to 0$. In this limit the imaginary part of the DSR1* goes to zero, because of its prefactor λ, while the real part reduces to the effective action (9) of the non-deformed scalar field. Also as in the κ-deformed case the real part is given by a sum of field frequencies and the imaginary part by a sum of squares of field frequencies. We are led to raise the question whether any deformation of the Poincaré algebra will have both real and imaginary parts and, if this is the case, it will follow the pattern of sum of frequencies for the real part and sum of squares of frequencies for the imaginary part.

Due to lack of charge conjugation symmetry in DSR1* theory its effective action reveals that stationary vacuum fluctuations obey the dispersion relations of the negative frequencies while the possible vacuum decay described by the imaginary part of the action occurs according to the dispersion relations of both negative and positive frequencies. Clearly, those features cannot be seen nor in the non-deformed theory neither in the κ-deformed theory, because the negative and positive frequencies in their

effective actions have the same modulus, due to charge conjugation symmetry. Besides the sole presence of negative frequencies in the real part, they appear in a rather peculiar way, namely: the unsaturated frequencies with the usual weight $1/2$ in its sum and the transaturated frequencies with just half of this weight, $1/4$.

We should also notice that in the imaginary part of the effective action there are contributions not only from the positive and negative frequencies, but also from the frequencies $\omega_n^{(>)}$ that we defined in (22). It is remarkable that the real quantity $\omega_n^{(>)}$ is not included into the theory by hand, but appears naturally in the course of calculation. With the presence of those transaturated positive frequencies in the imaginary part we may say that the whole range of momenta contributes to the imaginary part, with the sole exception of the saturation momentum $1/\lambda$.

Due to the presence of both real and imaginary parts in DSR1* theory it is open the possibility that the real part gives rise to a Casimir energy when boundary conditions are present and the imaginary part to a creation rate of field excitations even with static boundary conditions. However, it is not a straightforward task the calculation of those quantities, due to the abrupt cut in the summation of frequencies when saturation moment is reached, as it happens in the real and imaginary parts of DSR1* effective action (23). This is a problem that deserves careful analysis, since Casimir energy and creation rates associated to boundary conditions are useful theoretical probes into any theory. For example, a negative imaginary part in the renormalized effective action reveals a serious inconsistency in the theory, as remarked above.

Finally, we should make a remark about the lack of charge conjugation symmetry in DSR1*. Indeed, the overwhelming asymmetry between the quantities of matter and antimatter in the universe requires an asymmetry somewhere in the theory of its evolution. It has been argued [12] that the creation rate of field excitations (16) obtained from the imaginary part of the κ-deformed effective action provides a mechanism for the creation of matter and radiation in the early universe. Now, if a theory has not only an imaginary part in the effective action, but also a manifest charge conjugation asymmetry, it is open the possibility that the above mentioned mechanism of creation of matter and radiation occurs in a non-symmetrical way. This may provide an alternative theoretical setting to investigate the particle-antiparticle asymmetry in nature. As we have seen, DSR1* is an example of a theory with those features, although we cannot yet decide for sure what would be the correct theory describing such asymmetry in nature. Much more work is necessary to reach any conclusion on such matters, but it is worthwhile to consider the above remarks, due to the importance of the subject.

There are some natural continuations to this work. The obvious one is to calculate the Casimir energy and possible creation rate of field excitations from the DSR1* effective action. The result of this calculation may be very important to determine the consistency of the theory and its general features as, for example, if the presence of transaturated quantities in the effective action is spurious or not. If they are spurious we will have to face the question of how to eliminate them.

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